The Riemann zeta zeros from an asymptotic perspective

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Bernard Riemann (1826–66) was a student of Carl Friedrich Gauss (1777–1855) at the University of Gottingen in Lower Saxony. Since his youth, Gauss had conjectured that the distribution of the primes could approximately be given by

$$\frac{N}{\ln(N)},$$

the approximation becoming better as N became larger. In time, this conjecture became known as the *prime number conjecture* and then as the *prime number theorem* or PNT. Today we write this as

$$\pi(N) \sim \frac{N}{\ln(N)}$$

where $\pi(N)$ represents the number of primes less than a given number N. A second way to write the PNT is $\pi(x) \sim \mathrm{Li}(x)$ where $\mathrm{Li}(x)$ is called the log integral function and represents the definite integral of $\frac{1}{\ln(t)}$ as t varies from 2 to x.

In 1859, on the occasion of being elected as a corresponding member of the Berlin Academy, Bernard Riemann presented a lecture with the title *On the Number of Primes Less Than a Given Magnitude*, in which he presented a mathematics formula, derived from complex integration, which gave a precise count of the primes on the understanding that one of the terms in the formula, which depended on a knowledge of the non-trivial zeros of the zeta function, could be evaluated. Riemann had calculated some of the non-trivial zeros and found them all to have a real part equal to 0.5. He conjectured that every zero of the zeta function had a real part equal to 0.5. This became known as the *Riemann conjecture* which evolved into the *Riemann hypothesis* as more supporting evidence became available.

Riemann's 1859 paper can be found in Stephen Hawking's 2005 book *God Created the Integers*. Riemann's solution is given using complex number integration and is assessable to university students studying higher mathematics courses. A more assessable solution for school students who are studying advanced mathematics (e.g., Specialist Mathematics in the *Australian*

Curriculum: Mathematics) is given in John Derbyshire's very readable 2012 book *Prime Obsession.*

The aim of this paper is to show that the zeros of the Riemann zeta function

$$\zeta(s)=\sum_{k=1}^{\infty}\frac{1}{k^s},\text{ where } s=a+ib$$

all have the real part equal to one half.

This is demonstrated in two ways or methods. Each method shows that there can be only one value for the real part a, and since we know that there are zeros with $a = \frac{1}{2}$, then the proof is complete.

Notation

(a) The Riemann zeta function

$$\sum_{k=1}^{\infty}\left(\frac{1}{k^s}\right)=\frac{1}{1^s}+\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{4^s}+\ldots=\zeta(s)$$

where $s = a + ib$ and a and b are real.

(b) The alternating Riemann zeta function

$$\sum_{k=1}^{\infty}\left(\frac{1}{k^s}(-1)^{k-1}\right)=\frac{1}{1^s}-\frac{1}{2^s}+\frac{1}{3^s}-\frac{1}{4^s}+\ldots=\zeta_a(s)$$

where $s = a + ib$ and a and b are real.

(c) The symbol ~ between two functions or series means that the two are asymptotically equal to one another. One series might represent a convergent series in the variable x and the other might represent a divergent series in x but the two may be equal to one another to some degree, say four decimal places, when a finite number of terms of the divergent series is used. An example of this is given below:

$$\sum_{k=1}^{\infty}\frac{1}{k^2}\sim 1+\frac{B_0}{2^1}-\frac{B_1}{2^2}+\frac{B_2}{2^3}-\frac{B_3}{2^4}+\frac{B_4}{2^5}-\frac{B_5}{2^6}\ldots \qquad (*)$$

Now this is an example of an asymptotic series. We are approximating a slowly converging series with a diverging series. The numbers B_0, B_1, B_2, B_3, … represent the Bernoulli numbers $1, -\frac{1}{2}, \frac{1}{6}, 0, -\frac{1}{30}, 0, \frac{1}{42}, 0, -\frac{1}{30}, 0, \ldots$, with $B_{2k+1} = 0$ for $k \geq 1$.

The value of the right-hand side of (*) for varying numbers of non-zero terms is (to 6 decimal places):

One term:	1
Two terms:	1.5
Three terms:	1.625
Four terms:	1.645833
Five terms:	1.644792
Six terms:	1.64498
Seven terms:	1.644913
Eight terms:	1.64495

Now the exact value of the left-hand side of (*) is $\frac{\pi^2}{6}$ = 1.644934 (to 6 decimal places).

So we see that the right-hand side is equal to the left-hand side correct to four decimal places if seven terms of the right-hand side series are used. However, if more than 11 terms of the series on the right-hand side are used, then the two sides are no longer equal correct to four decimal places. This is mainly because the first eight Bernoulli numbers lie between zero and 1.1 in magnitude while the next eight Bernoulli numbers lie between 1.1 and 27 298 231.1 in magnitude, which far outpace the denominators of successive terms which are only increasing by a factor of 2.

Method one

In this method, in $s = a + ix$, a is considered constant and x is variable and $\zeta_a(a + ix)$ is found asymptotically in the form $(A_0 + A_1x + A_2x^2 + A_3x^3 + A_4x^4 + \ldots) + i(C_0 + C_1x + C_2x^2 + C_3x^3 + C_4x^4 + \ldots)$ where the A_i and C_i are real numbers.

Justification for assuming such a result is as follows:

$$\begin{aligned}\zeta_a(s) &= \left(1-2^{1-s}\right)\zeta(s)\\ &= \left(1-2^{1-s}\right)\sum_{k=1}^{\infty}\frac{1}{k^s}\\ &= \left(1-2^{1-s}\right)\sum_{k=1}^{\infty}\left(\frac{1}{k^a}\right)\left[\cos\left(x\ln(k)\right)-i\sin\left(x\ln(k)\right)\right]\end{aligned}$$

Now, $\cos(x\ln(k))$ and $\sin(x\ln(k))$ can be written in terms of power series in the variable x.

Also, $(1 - 2^{1-s}) = (1 - 2^{1-(a+ix)})$ can be written as a series in x with real coefficients.

Hence we have an infinite sum of terms of the form $S_1(x) + iS_2(x)$ multiplied by a series in x. This will give an expression in x with real coefficients. This is the proposal and justification.

However, the method to be used is not that outlined above. Instead, we equate $\zeta_a(s)$ and $(A_0 + A_1x + A_2x^2 + A_3x^3 + A_4x^4 + \ldots) + i(C_0 + C_1x + C_2x^2 + C_3x^3 + C_4x^4 + \ldots)$ with the understanding that they are asymptotically equal.

So, let

$$\begin{aligned}\zeta_a(s) = {}&(A_0 + A_1x + A_2x^2 + A_3x^3 + A_4x^4 + \ldots)\\ &+ i(C_0 + C_1x + C_2x^2 + C_3x^3 + C_4x^4 + \ldots) \qquad \text{(A)}\end{aligned}$$

We wish ultimately to find expressions for the coefficients A_i and C_i. Then we can look for solutions to $\zeta_a(s) = 0$ in the domain $0 < x < 1$.

So to begin:

$$\begin{aligned}&\frac{1}{1^{a+ix}}-\frac{1}{2^{a+ix}}+\frac{1}{3^{a+ix}}-\frac{1}{4^{a+ix}}+\ldots\\ &=\left(A_0+A_1x+A_2x^2+A_3x^3+A_4x^4+\ldots\right)+i\left(C_0+C_1x+C_2x^2+C_3x^3+C_4x^4+\ldots\right) \quad (1)\end{aligned}$$

Let $x = 0$ in equation (1)

$$\frac{1}{1^a} - \frac{1}{2^a} + \frac{1}{3^a} - \frac{1}{4^a} + \ldots = A_0 + iC_0$$

As the left-hand side is real, the right-hand side is real. So

$$C_0 = 0 \text{ and } A_0 = \frac{1}{1^a} - \frac{1}{2^a} + \frac{1}{3^a} - \frac{1}{4^a} + \ldots$$

Now use differentiation of equation (1) to find more coefficients. The first differentiation gives:

$$\begin{aligned} &i\left(\frac{(\ln(2)}{2^{a+ix}} - \frac{\ln(3)}{3^{a+ix}} + \frac{\ln(4)}{4^{a+ix}} - \ldots\right) \\ &= \left(A_1 + 2A_2x + 3A_3x^2 + \ldots\right) + i\left(C_1 + 2C_2x + C_3x^2 + \ldots\right) \qquad (2) \end{aligned}$$

Letting $x = 0$ in equation (2) we obtain

$$i\left(\frac{(\ln(2)}{2^a} - \frac{\ln(3)}{3^a} + \frac{\ln(4)}{4^a} - \ldots\right) = A_1 + iC_1$$

Equating real and imaginary parts, we find

$$A_1 = 0 \text{ and } C_1 = \frac{(\ln(2)}{2^a} - \frac{\ln(3)}{3^a} + \frac{\ln(4)}{4^a} - \ldots$$

Subsequent differentiation and setting $x = 0$ gives the following results:

$$\begin{aligned}
A_0 &= 1 - \left(\frac{1}{2^a} - \frac{1}{3^a} + \frac{1}{4^a} - \frac{1}{5^a} + \frac{1}{6^a} + \ldots\right) \\
C_0 &= 0 \\
A_1 &= 0 \\
C_1 &= \frac{\ln 2}{2^a} - \frac{\ln 3}{3^a} + \frac{\ln 4}{4^a} - \frac{\ln 5}{5^a} + \ldots \\
A_2 &= \frac{1}{2!}\left(\frac{\ln^2 2}{2^a} - \frac{\ln^2 3}{3^a} + \frac{\ln^2 4}{4^a} - \frac{\ln^2 5}{5^a} + \ldots\right) \\
C_2 &= 0 \\
A_3 &= 0 \\
C_3 &= -\frac{1}{3!}\left(\frac{\ln^3 2}{2^a} - \frac{\ln^3 3}{3^a} + \frac{\ln^3 4}{4^a} - \frac{\ln^3 5}{5^a} + \ldots\right) \\
A_4 &= -\frac{1}{4!}\left(\frac{\ln^4 2}{2^a} - \frac{\ln^4 3}{3^a} + \frac{\ln^4 4}{4^a} - \frac{\ln^4 5}{5^a} + \ldots\right) \\
C_4 &= 0 \\
A_4 &= 0 \\
C_5 &= \frac{1}{5!}\left(\frac{\ln^5 2}{2^a} - \frac{\ln^5 3}{3^a} + \frac{\ln^5 4}{4^a} - \frac{\ln^5 5}{5^a} + \ldots\right) \\
A_6 &= \frac{1}{6!}\left(\frac{\ln^6 2}{2^a} - \frac{\ln^6 3}{3^a} + \frac{\ln^6 4}{4^a} - \frac{\ln^6 5}{5^a} + \ldots\right)
\end{aligned}$$

In general, $A_n = 0$ if n is odd; $C_n = 0$ if n is even;

$$A_n = (-1)^{\frac{n}{2}+1}\left(\frac{1}{n!}\right)\sum_{k=2}^{\infty}(-1)^k \frac{\ln^n k}{k^a} \text{ if } n \text{ is even and } n \neq 0$$

$$C_n = (-1)^{\frac{n}{2}-1}\left(\frac{1}{n!}\right)\sum_{k=2}^{\infty}(-1)^k \frac{\ln^n k}{k^a} \text{ if } n \text{ is odd}$$

So the first aim has been achieved, that is, we have found $\zeta_a(a + ix)$ asymptotically in the form

$$(A_0 + A_1x + A_2x^2 + A_3x_3 + A_4x^4 + \ldots) + i(C_0 + C_1x + C_2x^2 + C_3x^3 + C_4x^4 + \ldots).$$

In fact, noting that some of the coefficients are zero, we can write:

$$\zeta_a(a+ix) \sim (A_0 + A_2x^2 + A_4x^4 + A_6x^6 + A_8x^8 + \ldots) + i(C_1x + C_3x^3 + C_5x^5 + C_7x^7 + \ldots)$$

Notice that the imaginary part of $\zeta_a(a + ix)$ has x as a factor.

Now let us go to our second aim which is to consider the equation $\zeta_a(a + ix) = 0$ for $0 < x < 1$.

So we have

$$(A_0 + A_2x^2 + A_4x^4 + A_6x^6 + A_8x^8 + \ldots) + i(C_1x + C_3x^3 + C_5x^5 + C_7x^7 + \ldots) = 0 + i.0$$

Equating real and imaginary parts, we have

$$(A_0 + A_2x^2 + A_4x^4 + A_6x^6 + A_8x^8 + \ldots) = 0 \text{ and } (C_1x + C_3x^3 + C_5x^5 + C_7x^7 + \ldots) = 0.$$

Let these two equations be rearranged as follows:

The first becomes

$$A_0\left(1+\frac{x^2A_2}{A_0}+\frac{x^4A_4}{A_0}+\frac{x^6A_6}{A_0}+\ldots\right)=0$$

and the second becomes

$$C_1x\left(1+\frac{x^2C_3}{C_1}+\frac{x^4C_5}{C_1}+\frac{x^6C_7}{C_1}+\ldots\right)=0$$

Dividing these last two equations by the non-zero numbers A_0 and C_1 respectively, we have

$$\left(1+\frac{x^2A_2}{A_0}+\frac{x^4A_4}{A_0}+\frac{x^6A_6}{A_0}+\ldots\right)=0 \text{ and } x\left(1+\frac{x^2C_3}{C_1}+\frac{x^4C_5}{C_1}+\frac{x^6C_7}{C_1}+\ldots\right)=0$$

Now let us consider solutions of $\zeta_a(a + ix) = 0$. These must also be solutions of the above two equations. Also we know that $x = 0$ is not a solution of $\zeta_a(a + ix) = 0$. So we conclude that the three equations

$$\left(1+\frac{x^2A_2}{A_0}+\frac{x^4A_4}{A_0}+\frac{x^6A_6}{A_0}+\ldots\right)=0,\ \left(1+\frac{x^2C_3}{C_1}+\frac{x^4C_5}{C_1}+\frac{x^6C_7}{C_1}+\ldots\right)=0,\ \zeta_a(a+ix)=0$$

have the same solutions and that since the first two series are monic, they are equal.

Equating coefficients of the terms in x, we have $\dfrac{A_2}{A_0}=\dfrac{C_3}{C_1}$.

Then it follows that $\dfrac{A_2}{A_0}-\dfrac{C_3}{C_1}=0$.

Note here that it should be remembered that the equal signs being used above are not actually equal signs but rather they represent approximations to a certain number of decimal places, or asymptotically equal signs.

Now let $y=\dfrac{A_2}{A_0}-\dfrac{C_3}{C_1}$

and using the values of the coefficients found previously, we have

$$y=\left(\frac{1}{2!}\right)\frac{\left(\frac{\ln^2 2}{2^a}-\frac{\ln^2 3}{3^a}+\frac{\ln^2 4}{4^a}-\frac{\ln^2 5}{5^a}+\ldots\right)}{\left(1-\frac{1}{2^a}+\frac{1}{3^a}-\frac{1}{4^a}+\frac{1}{5^a}-\frac{1}{6^a}+\ldots\right)}+\left(\frac{1}{3!}\right)\frac{\left(\frac{\ln^3 2}{2^a}-\frac{\ln^3 3}{3^a}+\frac{\ln^3 4}{4^a}-\frac{\ln^3 5}{5^a}+\ldots\right)}{\left(\frac{\ln 2}{2^a}-\frac{\ln 3}{3^a}+\frac{\ln 4}{4^a}-\frac{\ln 5}{5^a}+\ldots\right)}$$

Notice that y has been expressed in terms of one unknown, a. This can be shown on a graph using the software *Graphmatica*, but first the unknown must be replaced by the variable x. This graph is shown below for the domain $0 \le x \le 1$, that is, $0 \le a \le 1$. Each of the four series involved has been truncated to ten terms.

The graph shown in Figure 1 is as it would appear on a hard copy using the *Graphmatica* software (see Appendix 1 for a larger section of this graph).

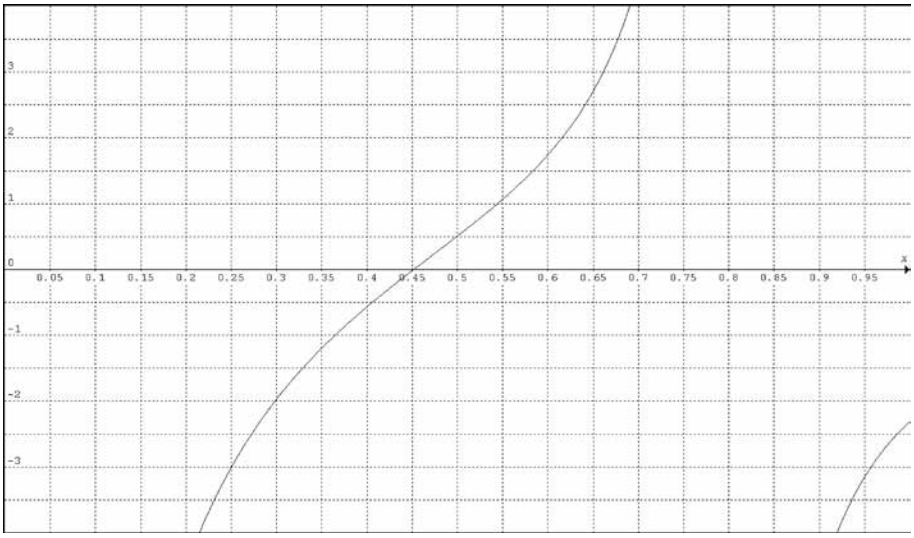


Figure 1.

Equations on screen:

y=.5*(f2(x))/(f1(x))+1/6*(h2(x))/(h1(x))

Functions used by these equations:

f=1

f1(x)=(2^–x)*(ln(2))^(2*f–2)–(3^–x)*(ln(3))^(2*f–2)
+(4^–x)*(ln(4))^(2*f–2)–(5^–x)*(ln(5))^(2*f–2)+(6^–x)*(ln(6))^(2*f–2)
–(7^–x)*(ln(7))^(2*f–2)+(8^–x)*(ln(8))^(2*f–2)–(9^–x)*(ln(9))^(2*f–2)
+(10^–x)*(ln(10))^(2*f–2)–(11^–x)*(ln(11))^(2*f–2)

f2(x)=(2^–x)*(ln(2))^(2*f)–(3^–x)*(ln(3))^(2*f)+(4^–x)*(ln(4))^(2*f)
–(5^–x)*(ln(5))^(2*f)+(6^–x)*(ln(6))^(2*f)
–(7^–x)*(ln(7))^(2*f)+(8^–x)*(ln(8))^(2*f)
–(9^–x)*(ln(9))^(2*f)+(10^–x)*(ln(10))^(2*f)–(11^–x)*(ln(11))^(2*f)

h1(x)=(2^–x)*(ln(2))^(2*f–1)–(3^–x)*(ln(3))^(2*f–1)
+(4^–x)*(ln(4))^(2*f–1)–(5^–x)*(ln(5))^(2*f–1)+(6^–x)*(ln(6))^(2*f–1)
–(7^–x)*(ln(7))^(2*f–1)+(8^–x)*(ln(8))^(2*f–1)
–(9^–x)*(ln(9))^(2*f–1)+(10^–x)*(ln(10))^(2*f–1)–(11^–x)*(ln(11))^(2*f–1)

h2(x)=(2^–x)*(ln(2))^(2*f+1)–(3^–x)*(ln(3))^(2*f+1)
+(4^–x)*(ln(4))^(2*f+1)–(5^–x)*(ln(5))^(2*f+1)
+(6^–x)*(ln(6))^(2*f+1)–(7^–x)*(ln(7))^(2*f+1)
+(8^–x)*(ln(8))^(2*f+1)–(9^–x)*(ln(9))^(2*f+1)
+(10^–x)*(ln(10))^(2*f+1)–(11^–x)*(ln(11))^(2*f+1)

In the following conclusion, use is made of the fact that $\zeta(s)$ and $\zeta_a(s)$ have the same zeros, that is, $\zeta(s) = 0$ and $\zeta_a(s) = 0$ have the same roots, e.g., $s = 0.5 + i14.134725$ approximately.

Conclusion for Method 1

From the graph we see that there is only one intercept in the domain $0 < x < 1$; this occurs at about $x = 0.45$. This means that there is only one possible value of a in the domain $0 < a < 1$ at about $a = 0.45$. Now it is known that all non–trivial Riemann zeta zeros have real part a lying in the interval $0 < a < 1$. But our graph shows that there is only one possible value for a in this interval. And since it is also known that there are Riemann zeta zeros with real part $a = 0.5$, then we conclude that the value of $a = 0.45$, shown on the above graph, is an approximation for $a = 0.5$. Then we finally conclude that if $s = a + ib$ is a solution to the equation $\zeta(s) = 0$ or $\zeta_a(s) = 0$ then $a = 0.5$.

Method 2

The second method begins with the asymptotic formula

$$\sum_{k=1}^{\infty}\left(\frac{1}{k^s}\right) \sim 1 + \frac{B_0}{2^{s-1}(s-1)} - \frac{B_1}{2^s} + \frac{B_2 s}{2!2^{s+1}} + \frac{B_4 s(s+1)(s+2)}{4!2^{s+3}} + \frac{B_6 s(s+1)(s+2)(s+3)(s+4)}{6!2^{s+5}} + \ldots \tag{B}$$

where B_i are the Bernoulli numbers and $s = a + ib$.

In this method, the intention is firstly to truncate the right-hand side of the above equation to two terms, let $s = a + ix$ as in Method 1, express the truncated series in the form $S_1(x) + iS_2(x)$, where

$$S_1(x) = (D_0 + D_1 x + D_2 x^2 + D_3 x^3 + D_4 x^4 + \ldots),$$
$$S_2(x) = (E_0 + E_1 x + E_2 x^2 + E_3 x^3 + E_4 x^4 + \ldots),$$

compare the coefficients of x^2 as in Method 1 and produce a graph. Secondly, truncate the right-hand side of (B) to three terms and do the same as above to produce a graph; then truncate the right-hand side to four terms, and then five terms, producing four graphs in total. Finally, a conclusion is arrived at in a similar vein to that in Method 1.

Step 1

Truncate the right-hand side of (B) to two terms and let $s = a + ix$.

$$\sum_{k=1}^{\infty}\left(\frac{1}{k^s}\right) \sim 1 + \frac{B_0}{2^{s-1}(s-1)} = 1 + \frac{B_0 2^{1-s}}{s-1} = 1 + \frac{B_0 2^{1-s}}{a+ix-1}$$
$$= 1 + B_0\left(e^{\ln 2}\right)^{1-s}\frac{(a-1)-ix}{(a-1+ix)(a-1-ix)}$$
$$= 1 + B_0\left(e^{\ln 2}\right)^{(1-a)-ix}\frac{(a-1)-ix}{(a-1+ix)(a-1-ix)}$$

Now multiply both sides by $[(a-1)^2 + x^2]$ to obtain

$$\left[(a-1)^2+x^2\right]\sum_{k=1}^{\infty}\left(\frac{1}{k^s}\right)\sim\left[(a-1)^2+x^2\right]+B_0 2^{1-a}\left[(a-1)-ix\right]\left[\cos(x\ln 2)-i\sin(x\ln 2)\right]$$

Writing $$\cos(x\ln 2)=1-\frac{x^2\ln^2 2}{2!}+\frac{x^4\ln^4 2}{4!}+\ldots$$

and $$\sin(x\ln 2)=x\ln 2-\frac{x^3\ln^3 2}{3!}+\frac{x^5\ln^5 2}{5!}-\ldots$$

we arrive at the following:

$$\left[(a-1)^2+x^2\right]\zeta_a(a+ix)$$

$$\sim\left\{\left[(a-1)^2+B_0 2^{1-a}(a-1)\right]+\left[1-B_0 2^{1-a}(a-1)\frac{\ln^2 2}{2!}-B_0 2^{1-a}\ln 2\right]x^2+\text{ terms in } x^4,\ x^6,\ \ldots\right\}$$

$$+i\left\{-B_0 2^{1-a}\left[1+(a-1)\ln 2\right]x+B_0 2^{1-a}\left[\frac{\ln^2 2}{2!}+\frac{(a-1)\ln^3 2}{3!}\right]x^3+\text{ terms in } x^5,\ x^7,\ \ldots\right\}$$

We note that the imaginary part of the above has a factor of x. When this is factorised, the result can be written in the form:

$$\left[(a-1)^2+x^2\right]\zeta_a(a+ix)\sim(D_0+D_2x^2+D_4x^4+\ldots)+ix\left(E_0+E_2x^2+E_4x^4+\ldots\right)$$

and then as

$$\left[(a-1)^2+x^2\right]\zeta_a(a+ix)\sim D_0\left(1+\frac{D_2x^2}{D_0}+\frac{D_4x^4}{D_0}+\ldots\right)+ixE_0\left(1+\frac{E_2x^2}{E_0}+\frac{E_4x^4}{E_0}+\ldots\right)$$

Using similar reasoning to that used in Method 1, we can say the following:

The equation

$$\left[(a-1)^2+x^2\right]\zeta_a(a+ix)=0$$

implies that

$$\left(1+\frac{D_2x^2}{D_0}+\frac{D_4x^4}{D_0}+\ldots\right)=0$$

and

$$\left(1+\frac{E_2x^2}{E_0}+\frac{E_4x^4}{E_0}+\ldots\right)=0$$

That

$$\left[(a-1)^2+x^2\right]\zeta_a(a+ix),\left(1+\frac{D_2x^2}{D_0}+\frac{D_4x^4}{D_0}+\ldots\right)\text{and}\left(1+\frac{E_2x^2}{E_0}+\frac{E_4x^4}{E_0}+\ldots\right)$$

have the same zeros;

That

$$\left(1+\frac{D_2}{D_0x^2}+\frac{D_4}{D_0x^4}+\ldots\right)=\left(1+\frac{E_2}{E_0x^2}+\frac{E_4}{E_0x^4}+\ldots\right)$$

Equating the coefficients of x^2 in the two series above gives

$$\frac{D_2}{D_0}=\frac{E_2}{E_0}$$

and consequently

$$\frac{D_2}{D_0}-\frac{E_2}{E_0}=0$$

As in Method 1, please note here that it should be remembered that the equal signs being used above are not actually equal signs but rather they represent approximations to a certain number of decimal places, or asymptotically equal signs.

Let $y = \frac{D_2}{D_0} - \frac{E_2}{E_0}$

Writing this out fully, we have:

$$y = \frac{\left[1 - B_0 2^{1-a}(a-1)\frac{\ln^2 2}{2!} - B_0 2^{1-a}\ln 2\right]}{\left[(a-1)^2 + B_0 2^{1-a}(a-1)\right]} - B_0 2^{1-a}\,\frac{\left[\frac{\ln^2 2}{2!} + \frac{(a-1)\ln^3 2}{3!}\right]}{-B_0 2^{1-a}\left[1+(a-1)\ln 2\right]}$$

This represents an equation in one unknown, a.
As mentioned previously, *Graphmatica* requires us to write equations in terms of x.

The graph of this equation using *Graphmatica* is shown in Figure 2 as the graph $y = \mathrm{h1}(x) - \mathrm{h2}(x)$.

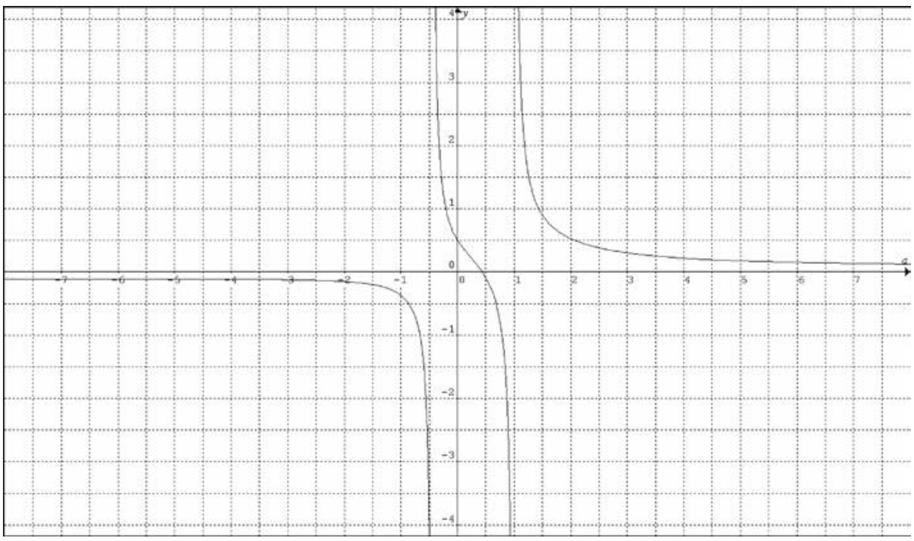


Figure 2

Equations shown in Figure 2:

$$y = \mathrm{h1}(x) - \mathrm{h2}(x)$$

Functions used by these equations:
f1(x)=(x–1)^2+b0*(x–1)*2^(1–x)
f2(x)=1–b0*(2^(1–x)*(x–1)*1/2*(ln(2))^2–b0*ln(2)*2^(1–x)
h1(x)=–f2(x)/f1(x)
f3(x)=1*b0*(2^(1–x))*(1+(x–1)*ln(2))
f4(x)=b0*(2^(1–x))*(1/2*(ln(2))^2+(x–1)*1/6*(ln(2))^3)
h2(x)=f4(x)/f3(x)

From the graph we see that there is only one intercept in the domain $0 < x < 1$; this occurs at about $x = 0.44$. That is, there is only one value of a in the domain $0 < a < 1$. As concluded in Method 1, this value of $a = 0.44$ is an approximation for $a = 0.5$.

Step 2

Truncate the right-hand side of (B) to three terms.
The method used in Step 1 above now results in the following equation and graph:

$$y = \frac{f2(a) + f6(a)}{f1(a) + f5(a)} - \frac{f4(a) + f8(a)}{f3(a) + f7(a)}$$

where

$$f1(a) = (a-1)^2 + b_0 2^{1-a}(a-1)$$

$$f2(a) = 1 - b_0 2^{1-a}\left(\frac{1}{2}(a-1)\ln^2 2 + \ln 2\right)$$

$$f3(a) = -b_0 2^{1-a}(1 + (a-1)\ln 2)$$

$$f4(a) = b_0 2^{1-a}\left(\frac{1}{2!}\ln^2 2 + \frac{1}{3!}(a-1)\ln^3 2\right)$$

$$f5(a) = -b_1 2^{-a}(a-1)^2$$

$$f6(a) = -b_1 2^{-a}\left(1 - \frac{1}{2!}(a-1)^2 \ln^2 2\right)$$

$$f7(a) = -b_1 2^{-a}(a-1)^2 \ln 2$$

$$f8(a) = -b_1 2^{-a}\left(\frac{1}{3!}(a-1)^2 \ln^3 2 - \ln 2\right)$$

From Figure 3, whose equation is $y = \mathrm{h}7(x) - \mathrm{h}8(x)$, we see that there are two values of a as intercepts. One at a = –0.1 approximately and the other at about $a = -1.3$. The important point to notice for this paper is that there is no value of a in the domain $0 < a < 1$.

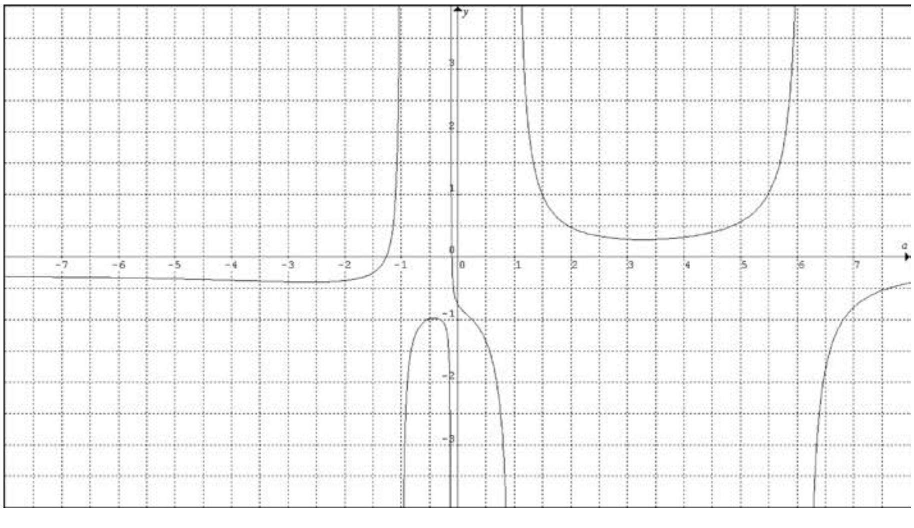


Figure 3

Step 3

Truncate the right-hand side of (B) to four terms.

This results in the graph shown in Figure 4 with the equation $y = \mathrm{h}13(x) - \mathrm{h}14(x)$. Here we find only one intercept at about $a = -0.07$. There is no value for a in $0 < a < 1$.

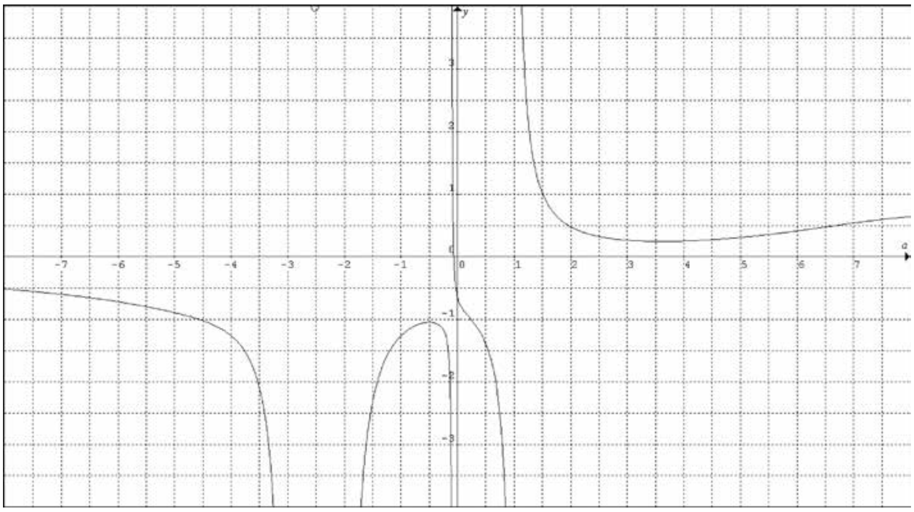


Figure 4

Step 4

Truncate the right-hand side of (B) to 5 terms and obtain the graph $y = \mathrm{h}19(x) - \mathrm{h}20(x)$ as shown in Figure 5. Here we find two intercepts at $a = -0.07$ and $a = -3.6$ but no values for a in $0 < a < 1$.

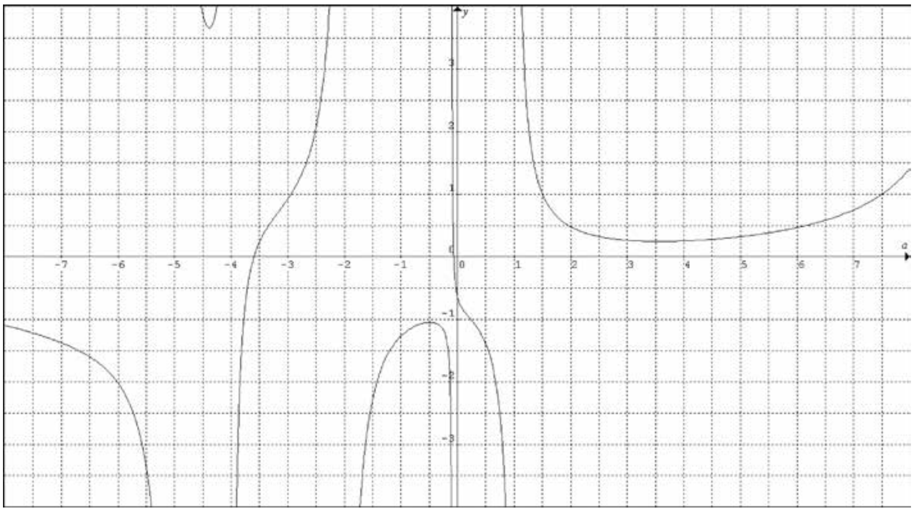


Figure 5

See Appendix 2 for the full list of equations used by *Graphmatica* in the graphs above.

Conclusion for Method 2

From the four graphs above we see that there is at most one intercept in the domain $0 < a < 1$; this occurs at about $a = 0.44$. Now it is known that all non-trivial Riemann zeta zeros have real part a lying in the interval $0 < a < 1$. Our graphs show that there is only one possible value for a in this interval. And since it is also known that there are Riemann zeta zeros with real part $a = 0.5$, then we conclude that the value of $a = 0.44$ is an approximation for $a = 0.5$. Then we finally conclude that if $s = a + ib$ is a solution to the equation $\zeta(s) = 0$ then $a = 0.5$.

Overall conclusion

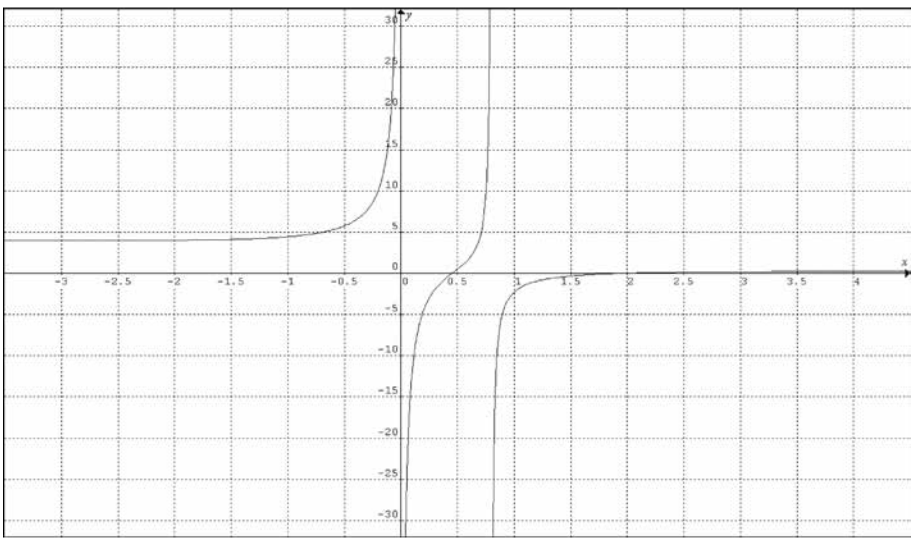
It has been shown that if $\zeta(a + ix) = 0$ then there can be only one value of a and the value of a has been shown to be of the order of 0.45. It has also been shown that since it is known that $s = 0.5 + i14.134725\ldots$ is one solution of the equation $\zeta(s) = 0$ then the real part of any Riemann zeta zero is $a = 0.5$.

Appendix 1

The following is a larger section of the graph used in Method 1 whose equation is:

$$y = \frac{1}{2!}\frac{\left(\frac{\ln^2 2}{2^a} - \frac{\ln^2 3}{3^a} + \frac{\ln^2 4}{4^a} - \frac{\ln^2 5}{5^a} + \ldots\right)}{\left((1 - \frac{1}{2^a} + \frac{1}{3^a} - \frac{1}{4^a} + \frac{1}{5^a} - \frac{1}{6^a} + \ldots\right)} + \frac{1}{3!}\frac{\left(\frac{\ln^3 2}{2^a} - \frac{\ln^3 3}{3^a} + \frac{\ln^3 4}{4^a} - \frac{\ln^3 5}{5^a} + \ldots\right)}{\left(\frac{\ln 2}{2^a} - \frac{\ln 3}{3^a} + \frac{\ln 4}{4^a} - \frac{\ln 5}{5^a} + \ldots\right)}$$

and its *Graphmatica* equation is y=.5*(f2(x))/(f1(x))+1/6*(h2(x))/(h1(x)).



y=.5(f2(x))/(f1(x))+1/6*(h2(x))/(h1(x))*

Appendix 2

Graphmatica equations used by the graphs in Method 2.

```
b0=1
b1=–1/2
b2=1/6
b4=–1/30
f1(x)=(x–1)^2+b0*(x–1)*2^(1–x)
f2(x)=1–b0*(2^(1–x))*(x–1)*1/2*(ln(2))^2–b0*ln(2)*2^(1–x)
h1(x)=f2(x)/f1(x)
f3(x)=–1*b0*(2^(1–x))*(1+(x–1)*ln(2))
f4(x)=b0*(2^(1–x))*(1/2*(ln(2))^2+(x–1)*1/6*(ln(2))^3)
h2(x)=f4(x)/f3(x)
f5(x)=–1*b1*(2^(–x))*(x–1)^2
f6(x)=–1*b1*(2^(–x))*(1–(x–1)^2*.5*(ln(2))^2)
f7(x)=–1*b1*(2^(–x))*ln(2)*(x–1)^2
f8(x)=–1*b1*(2^(–x))*(1/6*(x–1)^2*(ln(2))^3–ln(2))
h3(x)=f1(x)+f5(x)
h4(x)=f2(x)+f6(x)
h5(x)=f3(x)+f7(x)
h6(x)=f4(x)+f8(x)
h7(x)=h4(x)/h3(x)
h8(x)=h6(x)/h5(x)
f9(x)=b2*1/2*(2^(–x–1))*x(x–1)^2
f10(x)=b2*1/2*(2^(–x–1))*(–1*x*(x–1)^2*(ln(2))^2*1/2
     +ln(2)*(x–1)^2+x)
f11(x)=b2*1/2*(2^(–x–1))*(–1*x*(x–1)^2*ln(2)+(x–1)^2)
f12(x)=b2*1/2*(2^(–x–1))*(1/6*x*(x–1)^2*(ln(2))^3
     –1/2*(x–1)^2*(ln(2))^2–x*ln(2)+1)
h9(x)=h3(x)+f9(x)
h10(x)=h4(x)+f10(x)
h11(x)=h5(x)+f11(x)
h12(x)=h6(x)+f12(x)
h13(x)=h10(x)/h9(x)
h14(x)=h12(x)/h11(x)
f13(x)=b4*1/24*(2^(–x–3))*(x–1)^2*(x^3+3*x^2+2*x)
f14(x)=b4*1/24*(2^(–x–3))*(–3*(x–1)^2*(x+1)
     –1/2*((ln(2))^2)*(x^3+3*x^2+2*x)*(x–1)^2
     +ln(2)*(3*x^2+6*x+2)*(x–1)^2+(x^3+3*x^2+2*x))
f15(x)=b4*1/24*(2^(–x–3))*((3*x^2+6*x+2)*(x–1)^2
     –ln(2)*(x^3+3*x^2+2*x)*(x–1)^2)
```

f16(x)=b4*1/24*(2^(−x−3))*(−1*(x−1)^2+3*ln(2)*(x+1)*(x−1)^2
+((x−1)^2)*1/6*(x^3+3*x^2+2*x)*(ln(2))^3
−((ln(2))^2)*1/2*(3*x^2+6*x+2)*(x−1)^2+(3*x^2+6*x+2)
−ln(2)*(x^3+3*x^2+2*x))

h15(x)=h9(x)+f13(x)

h16(x)=h10(x)+f14(x)

h17(x)=h11(x)+f15(x)

h18(x)=h12(x)+f16(x)

h19(x)=h16(x)/h15(x)

h20(x)=h18(x)/h17(x)

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